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Classical closure theory and Lam's interpretation of ϵ–renormalization group theory

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It is shown that Lam's formulation of renormalization group theory [Phys. Fluids A **4**, 1007 (1992)] is essentially the physical space version of the spectral classical closure theory [Leslie and Quarini, J. Fluid Mech. **91**, 65 (1979)].

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In this Brief Report, we demonstrate that Lam's ϵ–renormalization group theory (RNG) model [1] is essentially the physical space version of the classical closure theory [2] in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter. The incompressible Navier-Stokes (NS) equations are

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{1}{\rho}\nabla p+\nu_0\nabla^2\mathbf{v} , \tag{1}$$

where ν_0 is the molecular viscosity, ρ is the density, and p is the pressure and can be determined from (1) using $\nabla\cdot\mathbf{v}=0$. The external driving force that sustains the turbulence and which acts in the very small wave-number region is not included in (1) since it plays no part in the energy cascade process in the inertial range [2].

As in both closure and RNG theories, the velocity field is filtered into two components,

$$\mathbf{v}=\mathbf{v}^{<}+\mathbf{v}^{>},\quad p=p^{<}+p^{>} , \tag{2}$$

where the Fourier-transformed fields are

$$v_i^{<}(\mathbf{k},t)=G(k)v_i(\mathbf{k},t) , \tag{3}$$

$$v_i^{>}(\mathbf{k},t)=[1-G(k)]v_i(\mathbf{k},t) . \tag{4}$$

The sharp cutoff filter of classical closure theory is exactly the same as the RNG technique of separating the subgrid from the resolvable scales at the cutoff wave number Λ,

$$G(k)=\begin{cases}0 & \text{if } k>\Lambda\\ 1 & \text{if } k<\Lambda .\end{cases} \tag{5}$$

In the classical closure theory of Leslie and Quarini (LQ) [2], the filtered NS equation is

$$\left[\frac{\partial}{\partial t}+[\nu_0+\nu_d(k)]k^2\right]v_\alpha^{<}(\mathbf{k},t) =M_{\alpha\beta\gamma}(k)\int d\,\mathbf{p}\,d\mathbf{q}\,v_\beta^{<}(\mathbf{p},t)v_\gamma^{<}(\mathbf{q},t)+f_\alpha(\mathbf{k},t) , \tag{6}$$

where $M_{\alpha\beta\gamma}(k)$ is the standard nonlinear coupling coefficient [2,3]. For convenience, we have added to *both* sides a wave-number dependent turbulent eddy viscosity $\nu_d(k)$, which is at the moment unspecified. The term $f(k,t)$ accounts for the Reynolds stress [2,4],

$$R_{\beta\gamma}\equiv v_\beta^{>}(\mathbf{p},t)v_\gamma^{>}(\mathbf{q},t) , \tag{7}$$

the cross stress [2,4]

$$C_{\beta\gamma}\equiv v_\beta^{<}(\mathbf{p},t)v_\gamma^{>}(\mathbf{q},t)+v_\beta^{>}(\mathbf{p},t)v_\gamma^{<}(\mathbf{q},t) , \tag{8}$$

and the added eddy viscosity $\nu_d(k)$

$$f_\alpha(\mathbf{k},t)\equiv\nu_d(k)k^2v_\alpha^{<}(\mathbf{k},t) +M_{\alpha\beta\gamma}(k)\int d\mathbf{p}\,d\mathbf{q}[C_{\beta\gamma}+R_{\beta\gamma}] . \tag{9}$$

In (7) and (8), $|\mathbf{p}+\mathbf{q}|<\Lambda$. It is important to realize that no random force has been inserted here.

In the Lam approach to ϵ–RNG [1], one works in physical space rather than wave-number space. The exact resolvable scale Navier-Stokes equations can be written

$$\left[\frac{\partial}{\partial t}-(\nu_0+\nu_T)\nabla^2\right]\mathbf{v}^{<}=-\frac{1}{\rho}\nabla p^{<}-\nabla\cdot(\mathbf{v}^{<}\mathbf{v}^{<})+\mathbf{g}^{\text{fast}} , \tag{10}$$

where $\mathbf{g}^{\text{fast}}$ is defined by

$$\mathbf{g}^{\text{fast}}=\nabla\cdot(\mathbf{v}^{<}\mathbf{v}^{<}-\mathbf{v}\mathbf{v})-\nu_T\nabla^2\mathbf{v}^{<} =-\nabla\cdot(2\mathbf{v}^{>}\mathbf{v}^{<}+\mathbf{v}^{>}\mathbf{v}^{>})-\nu_T\nabla^2\mathbf{v}^{<} . \tag{11}$$

Note that Lam has introduced a *k-independent* turbulent eddy viscosity ν_T, which remains to be chosen. g^{fast} is generated by the filtering process. The term $\mathbf{g}^{\text{fast}}$ in physical space corresponds to the term $\mathbf{f}(\mathbf{k},t)$ in wave-number space, in Eq. (9).

The classical theory proceeds from this point by the use of certain "closure approximations" [2,3]. An equation for the resolvable spectral energy, $\bar{E}(k,t)$, can readily be derived,

$$\left[\frac{\partial}{\partial t}+2\nu_0k^2\right]\bar{E}(k,t)=\bar{T}(k,t)+T^{>}(k,t) , \tag{12}$$

where $\bar{T}(k,t)$ is the resolvable scale energy transfer and $T^{>}(k,t)$ is the energy transfer caused by the cross and Reynolds stresses [2], which can be put into the form [2,5]

$$T^{>}(k,t)\equiv-2\nu_d(k)k^2\bar{E}(k,t)+U(k) . \tag{13}$$

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$U(k)$, which represents the backscatter of energy from small to resolvable scales and is also the spectrum of the correlation function of $\mathbf{f}$, is given by

$$U(k)\equiv\int_\Delta dp\, dq\, B(k,p,q)E(p)E(q)G^2(k)\times[1-G(p)G(q)]\ . \qquad (14)$$

$\nu_d(k)$, the *drain eddy viscosity*, is given by

$$\nu_d(k)\equiv\int_\Delta dp\, dq\, A(k,p,q)E(q)[1-G(p)G(q)]\ . \qquad (15)$$

The integration domain is denoted by the expression Δ in which p and/or $q>\Lambda$. The explicit functional forms of A and B appearing in (14) and (15) are given in Leslie [3] and LQ [2].

Instead of trying to compute $\mathbf{g}^{\text{fast}}$ using closure approximations, Lam [1] simply tries to model its correlation function based on physical arguments. In his view, $\mathbf{f}$ is simply a guess of what $\mathbf{g}^{\text{fast}}$ should be for $k\approx\Lambda$ in the resolvable scale Navier-Stokes equation. He noted that in the absence of $\mathbf{f}$, the energy spectrum of the flow, computed from (6) driven by initial and/or boundary conditions will have a Kolmogorov dissipation wave number substantially smaller than Λ. The primary role of $\mathbf{f}$ is to extend for the resolvable scale velocity field the inertial range with a guaranteed Kolmogorov scaling for $k\approx\Lambda$ and beyond.

The forcing function in classical closure theory arises from filtering at the small scales. In modeling the correlation function of $\mathbf{f}$, Lam [1] *assumes* the form

$$\langle f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega')\rangle=\frac{2}{\Pi_3}\mathcal{E}\frac{1}{\Lambda^{4-\epsilon}}k^{-d+4-\epsilon}(2\pi)^{d+1}\times P_{ij}(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\ , \qquad (16)$$

where ω is frequency, $\mathcal{E}$ is the dissipation rate, d is the dimension of the physical space, Π_3 is a constant, and $P_{ij}(k)=\delta_{ij}-k_ik_j/k^2$. A multiplicative factor involving $\Lambda^{4-\epsilon}$ is introduced to maintain dimensional consistency for arbitrary ϵ. It is of some interest to compare Eq. (16) with the forcing correlation function introduced by Yakhot and Orszag (YO) [6]

$$\langle f_i(\mathbf{k},\omega)f_j(\mathbf{k}',\omega')\rangle=\frac{2}{\Theta}\mathcal{E}k^{-d+4-\epsilon}(2\pi)^{d+1}\times P_{ij}(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\ , \qquad (17)$$

where Θ is a known constant determined by $2D_0S_d/(2\pi)^{d+1}=1.594\mathcal{E}$ (YO [6]) and S_d is the area of a d-dimensional unit sphere. This form [7] is assumed to arise from forcing at $k=0$:

$$\langle ff\rangle=\delta(k)\mathcal{E}\delta(\mathbf{k}+\mathbf{k}')\ , \qquad (18)$$

with the use of Gel'fand's δ-function representation in the limit of $\epsilon\to4$ and $k\to0$,

$$\delta(k)=\lim_{\epsilon\to4}(4-\epsilon)k^{1-\epsilon}\quad\text{for } k\to0\ . \qquad (19)$$

To recover (17), it appears that (19) needs to be applied for $k\neq0$, without the $(4-\epsilon)$ factor.

Lam pointed out that the forcing correlation function, Eq. (16), should peak around Λ; its magnitude should be small for small k by an appropriate choice of ν_T; and its behavior for $k\gg\Lambda$ is unimportant and irrelevant for the evolution of the resolved modes. Most importantly, the correlation function now depends on Λ, while in ϵ–RNG [6,7], the correlation function is assumed to be "scale invariant." The dimensionless parameter ϵ in the correlation function is now available as a freely adjustable parameter, and Lam used it to make the "predicted value" of Kolmogorov constant acceptable. He showed that either $\epsilon=0$ or $\epsilon=0.923$ yields good results.

The stochastic backscatter $\mathbf{f}$, for isotropic homogeneous turbulence in three dimensions, has a k^4 spectrum to lowest order in wave number k (e.g., Ref. [5]). Specifically,

$$U(k)=\tfrac{14}{15}k^4\int_\Lambda^\infty dp\,\theta_{k,p,q}(t)\frac{[E(p)]^2}{p^2}\quad\text{for } k\to0\ , \qquad (20)$$

where $\theta_{k,p,q}(t)=1/[\mu_{k,p,q}(t)+\nu_0(k^2+p^2+q^2)]$ and $\mu_{k,p,q}(t)$ is an eddy-damping rate of the third-order moments associated with the wave vectors $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$. Thus, Lam's postulate (which was based on intuitive physical arguments) that $U(k)$ is small for small k is consistent with classical closure theory.

The advantage of the classical theory is that the energy equation is always satisfied and no restriction on the magnitude of Λ is imposed—so long as Λ is in the inertial range. On integrating (12) with respect to k for $0<k<\Lambda$, we obtain

$$\frac{\partial K}{\partial t}=\overline{\Pi}-\mathcal{E}\ , \qquad (21)$$

where K is the integral of $\overline{E}(k)$ over the resolved wave numbers, and $\mathcal{E}$ is defined by

$$\mathcal{E}\equiv\int_0^\Lambda T^>(k)dk=\int_0^\Lambda 2k^2\nu_n(k)\overline{E}(k)dk\ . \qquad (22)$$

$\overline{\Pi}$, the resolved energy transfer term, is given by

$$\overline{\Pi}\equiv\int_0^\Lambda\overline{T}(k)dk\ .$$

The *net eddy viscosity*, $\nu_n(k,t)$, is defined [2,5,8,9] as

$$\nu_n(k)\equiv\nu_d(k)-\nu_b(k) \qquad (23)$$

and $\nu_b(k,t)$, the *backscatter viscosity*, is given by

$$\nu_b(k)\equiv U(k)/[2k^2\overline{E}(k)]\ . \qquad (24)$$

From (15) and (24), one can show [10] that for k in the inertial range and $k\ll\Lambda$, the ratio of $\nu_b(k)$ to $\nu_d(k)$ is equal to $\frac{14}{15}(k/\Lambda)^{11/3}$. Spectral large-eddy simulations (LES) of Lesieur [5] and Lesieur and Rogallo [11] was based on the resolvable scale Navier-Stokes equation

$$\left[\frac{\partial}{\partial t}+[\nu_0+\nu_n(k)]k^2\right]v_\alpha^<(\mathbf{k},t)=M_{\alpha\beta\gamma}(k)\int\int d\mathbf{p}\,d\mathbf{q}\,v_\beta^<(\mathbf{p},t)v_\gamma^<(\mathbf{q},t)\ . \qquad (25)$$

Lam emphasized that $\mathcal{E}$, the energy dissipation rate of the turbulent flow in question, must be related to the parameters of the turbulent eddies by an *ad hoc* postulate under his formulation. Lam's choice [1] is

$$\mathcal{E}_L = \lim_{\Lambda\to\infty} 2\nu_T(\Lambda)\int_0^{\Lambda} k^2 E(k)dk \ . \tag{26}$$

The large Λ limiting process in (26) is needed to ensure that the dissipation rate can be adequately evaluated using information available from the resolved modes alone. In Lam's approach, the value of Λ must be sufficiently large such that the dissipation function $\mathcal{E}_L$ as given by (26) is independent of Λ. In physical variables, $\mathcal{E}_L$ is defined by

$$\mathcal{E}_L \equiv \nu_T(\Lambda)\left[\frac{\partial u_i^<}{\partial x_k}\right]^2 . \tag{27}$$

The Smagorinsky result for ν_T is recovered if $\mathcal{E}_L$ is eliminated between (27) and $\nu_T(\Lambda)=C_\nu \mathcal{E}_L^{1/3}\Lambda^{-4/3}$. In LES, the Lam requirement that Λ must be large enough is equivalent to requiring that (27), computed using data only from resolved modes, be "grid size" independent. In Lam's view, an LES calculation must exhibit a Kolmogorov spectrum using the resolved modes such that the limiting process in (26) is respected. If it does not, then the calculation would have no theoretical standing. Physically, if Λ is sufficiently large (so that $\mathcal{E}_L$ is independent of Λ), the contribution of back scattering to the dissipation would be negligible. The random force **f**, the *surrogate* of the $\mathbf{g}^{\text{fast}}$, does not appear explicitly in the final LES model of Lam: One needs only to provide a profile of $\langle \mathbf{ff} \rangle$ so as to introduce the adjustable parameter ϵ used in computing ν_T.

Thus, we find that Lam's formulation of ϵ–RNG [1] is essentially the physical space version of the spectral classical closure theory [2] with $\nu_n(k)$ being replaced by a phenomenological k-independent ν_T, but which now depends on arbitrary parameter ϵ.

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